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Stability of disturbance waves in developing shear flows: A review of 'ad-hoc' methods

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Abstract

This paper attempts to induce some formalism in the study of stability of developing shear flows, by use of so called 'ad-hoc' methods wherein all terms upto and inclusive of a particular order are lumped together in the same equation. However, whether the 'ordering' is perturbation or asympotic is somewhat vague. The classical example of stability of a developing shear flow is the stability problem of boundary-layer flow over a flat-plate including the non-parallel effects. Other examples are free shear flows; also, problems involving flow over alternate rigid and porous panels with suction, or suction from one wall and injection from the opposite wall. The quick method for stability analysis is to use the local Orr-Sommerfeld solution, using the quasi-parallel approximation. This is sometimes augmented by inclusion of some non-parallel terms, perhaps in an ad-hoc manner, and so called 'improved solutions' are obtained. The present paper attempts to establish a formal framework for these methods so that these methods are no more termed 'ad-hoc'.

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1. Introduction

This is a review of Gaster [1] and similar methods and Herbert's PSE c.f. Bertolotti et al[2] (hereafter referred to as BHS) for stability analysis in developing shear flows. Such methods are called 'ad-hoc' (AH) methods, a term we shall define now. In the present and similar related contexts, the ad-hoc method is a solution procedure that has the following attributes: (i) the solution is not exact; (ii) the procedure recognises the existence of perturbation, if not asymptotic, scales in the problem; and, whether the 'ordering' is perturbation or asymptotic in such problems, is vague; (iii) the procedure retains *all terms upto a particular (asymptotic or perturbation) order*, and excludes higher order terms; and, (iv) the most contentious issue in the procedure is that *all terms upto a given order are lumped together in the same equation*, which equation is later solved by a numerical method. It is the last attribute, '(iv)' above, which is the object of concern regarding the rationality of the procedure; especially for a methodology that has claims to being generic in applicability.

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Also, we discuss the method of matched asymptotic expansions (MAE), based on multi deck theories. Next, we have the method of direct numerical simulation (DNS) of the Navier-Stokes equations. Further, we have the experimental results.

Next we give a review of non parallel effects in the stability of the flat-plate boundary layer. The review is contextual, and mainly focuses on aspects related to the points raised afore. Out of the various earlier attempts in solving this problem, the first work to consider non-parallel effects was that of Barry and Ross [3]. However they did not keep all terms upto a given order in their analysis. They did observe some differences in the neutral curve due to non-parallel effects. Later, pioneering contributions were made, by so called 'ad-hoc' approaches, by Bouthier [5], and Gaster[1]. Important amongst the latter works are the ones by Bertolotti, Herbert and Spalart [2] who introduced the concept of the 'parabolised stability equations' (PSE), and, by Govindarajan and Narasimha [5] (hereafter referred to as GN). The GN method is another version of PSE using the similarity variable for which reason its versatility as a generic tool is limited as compared to PSE.

We would expect that the DNS results, and also (properly performed) experimental results, would provide the benchmarks for comparison with the respective results based on AH methods and MAE methods. Here we have an enigma: The AH methods are of debatable mathematical rigour; and yet results based on these match the benchmarks very well. Whereas, the MAE methods are sound in their mathematical rigour; yet the results based on these apparently do not meet the benchmarks. We offer a few comments on these last two statements. In the opinion of the present authors, the MAE methods should not be expected to match the "benchmark" of non-parallel neutral curves. MAE theories are proper asymptotic theories for the Reynolds number $R \to \infty$ and these theories do predict well the behaviour for the upper and lower branches of the neutral curve for $R \to \infty$. By contrast the AH methods are expected to operate at *finite and numerically large* values of R. Hence it is high time that one stopped looking at AH methods as asymptotic theories for $R \to \infty$. By the same token, the MAE methods, being rigorous asymptotic theories for $R \to \infty$, should not be expected to predict the behaviour at *finite* R, that is, for example, near the nose of the neutral curve. Given the complementarity, and not competition, between the MAE and AH methods, what remains is to establish a mathematical framework under which the AH methods can be justified, at least with some degree of rigour. If this is done successfully the AH methods would be in a position to shed the attribute "ad-hoc".

Actually, Gaster's work, and those of BHS and GN are the best, and fully correct, examples of the AH method. At the moment "correctness" is cited based on agreement with the DNS results of Fasel and Konzelman [6], and for other more concrete reasons which will become apparent later as the discussions proceed. However Gaster[1]and GN methods are tailored only to the flat-plate boundary layer problem. On the other hand the PSE method has gone very much forward and today it is a generic tool used for the stability analysis of many problems. Nevertheless, as the first successful proponent, Gaster[1] can be credited with having pioneered the AH methodology. Figure 1 shows the comparison between the results of Gaster[1], Govindrajan and Narasimha[5] and present results, with the benchmark results of Fasel and Konzelman [6].

Experimental work on the stability of the flat-plate boundary-layer flow problem was pioneered by Schubauer and Skramstad [7]. This was followed by Ross, Barnes, Burns and Ross [8], Babenko and Kozlov [9], and Kachanov, Kozlov and Levchenko [10]. All these works showed some mismatch with the results of Gaster[1], in that, the neutral curve extended to higher values of the frequency parameter $F = \left(\frac{\beta}{R}\right) \times 10^6$, where '$\beta$' is the temporal frequency and 'R' is the Reynolds number), than predicted by Gaster[1]. Ironically enough, some of the 'ad-hoc' non-parallel works, like that of Saric and Nayfeh [11], which were later on found to be not fully correct (vide BHS), showed good agreement with the earlier experimental results.

The stability of the flat-plate boundary layer problem found its final benchmark in the landmark results of Fasel and Konzelmann [6], obtained by direct numerical simulation (DNS). These results can be used as standard and final. Remarkably enough the results of Gaster[1], and of BHS and GN, agreed completely with the DNS results. The DNS results also agreed very well with the properly performed experimental results of Klingmann *et al.* [12]. These developments strongly suggest that, one should enquire as to why the 'ad-hoc' methods work; and, do such ad-hoc procedures have generic validity? Can the ad-hoc methods be put on a proper mathematical framework?

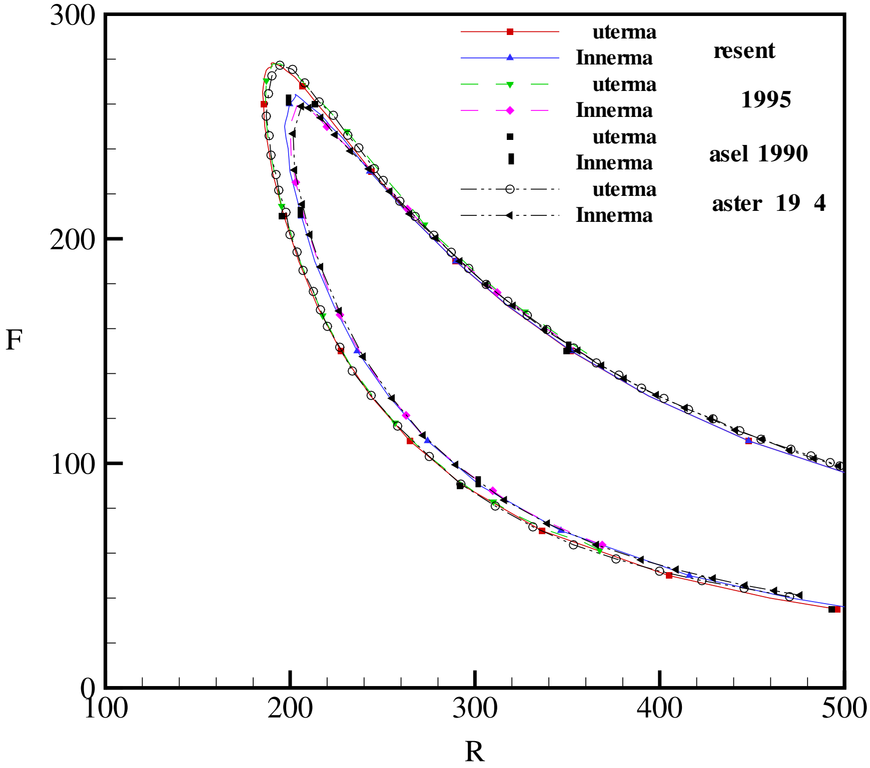

Fig. 1. Comparison of the present work with that of Govindarajan and Narasimha[5], Fasel and Konzelmann [6] and Gaster [1] based on inner and outer maximum. Outermax: Outer maximum of $\hat{u}_{rms}$; Innermax: Inner maximum of $\hat{u}_{rms}$

2. Dissection of the ad-hoc methodology

2.1. Background discussions

We begin by looking at the the classical form of the well known Orr-Sommerfeld (OS) equation given as follows:

$$L_{OS}(\phi) = 0 \quad ; \tag{1a}$$

where, ϕ is the disturbance streamfunction, and L_{OS} is the classical Orr-Sommerfeld operator given as,

$$L_{OS} \ = \ i\alpha\left[(\overline{u} - c)(D^2 - \alpha^2) - \overline{u}''\right] - \frac{1}{R}\left(D^4 - 2\alpha^2 D^2 + \alpha^4\right) \quad . \tag{1b}$$

In the above, $\overline{u}$ is the x-wise mean velocity (with overbar $(\bar{\ })$ generically indicating mean quantities), α is the spatial wave number, β is the temporal frequency, $c = \beta/\alpha$ is the phase speed, $R = \frac{UL}{\nu}$ is the Reynolds number with U and L as suitable velocity and length scales respectively, and, $D^n = d^n/dy^n$, and primes($'$) denote differentiation with respect to y. The disturbance amplitude function ϕ is a function of y only. This equation is the exact stability equation, for a given Fourier mode, in channel flow. Other near-parallel flow situations use this equation as a quasi-parallel approximation.

A numerical solution, if the grid is fine enough to have accounted for the detailed structure of the internal boundary layers in the OS equation, *prima facie* is a reliable method of solution. This, by no means implies that all the asymptotics can be ignored. For example, in the viscous group of terms in the OS equation $(1a, b)$, it is well known, by a simple application of the Tollmien critical layer scale, that in the critical layer

$$\frac{1}{R}\phi'''' \sim \alpha\overline{u}_c'(y - y_c)\phi'' \approx O(R^{\frac{1}{3}})\phi \ ; \quad \frac{1}{R}\alpha^2\phi'' \sim O(R^{-\frac{1}{3}})\phi \ ; \quad \frac{1}{R}\alpha^4\phi \sim O(R^{-1})\phi \ ; \tag{2a, b, c}$$

where subscript ‘c’ refers to the critical point. It can be shown that, whilst the largest order viscous term, viz. $\frac{1}{R}\phi''''$, is of the same order as the inviscid terms in the critical and wall layers, the other two viscous terms, viz. $\frac{1}{R}\alpha^2\phi''$ and $\frac{1}{R}\alpha^4\phi$, are *everywhere smaller* than the inviscid group of terms. Therefore, some justification is needed if such terms are to be kept, in a lumped manner, in the *same* equation, even if a numerical solution is being used. In fact, as we shall see presently that, the AH method is replete with terms of different orders being lumped in the same equation. This is quite in contrast with the rational hierarchy of equations developed by MAE.

2.2. Constant mean flows and quasi-constant mean flows

We need to look at a few flow situations where the mean flow field does not change with the flow direction x, and the respective stability equations corresponding to these. Following this , we will discuss the cases of quasi-constant flows.

2.2.1. Constant mean flows

The following are examples of constant mean flows:

(i) The first example is fully developed parallel channel flow or plane Poiseuille flow where the mean flow is perfectly parallel. The stability equation becomes exact and is given by eq. $(1a, b)$.

(ii) The second example is asymptotic suction in boundary layer flow. This flow is not strictly parallel in the physical sense, due to the presence of cross-flow, but this flow can be called ‘constant’. Here again we have an exact stability equation which is given as follows:

$$L_{OS}\phi + \bar{v}(\phi''' - \alpha^2\phi') = 0\,, \tag{3}$$

where $\bar{v}$ is the mean cross-flow velocity in the y-direction and is given by $\bar{v} = \bar{v}_w = -\frac{1}{R}$ (where subscripts ‘w’ is for the wall), and, R is based on the displacement thickness in this problem.

(iii) The third example is flow between parallel plates with injection through one plate and equal suction from the opposite plate. This flow is also a constant flow with the stability equation given by eq. (3), with $\bar{v} = \bar{v}_w = \pm\frac{R_w}{R}$ (+ for injection in the positive y direction, and − for suction in the negative y direction). Also, R_w is the cross flow Reynolds number given as $R_w = \frac{VL}{\nu}$, where V is the cross flow velocity. Also, R_w is numerically of order 1, with $R_w << R$.

Equation (3) is an extended version of the Orr-Sommerfeld equation, and, such equations may be called “Orr-Sommerfeld like” equations.

At this stage we need to bring into focus some relevant points. First that, numerical solutions of eqs. (1) and (3) are convenient, and nowadays can be taken to high levels of accuracy. Second that, solutions can be obtained over a wide range of values of α and R. Third that, changing the Reynolds number in such flows (e.g. mathematically we are so used to saying $R \to \infty$) is a bit involved in engineering terms; usually it means changing the flow conditions. If the geometry remains unchanged then it means changing the magnitude of the mean flow velocity. Fourth that, in engineering problems, the interest may centre around a wide range of R, which may not approximate the limit $R \to \infty$ very well; for example for flows near the nose of the neutral curve in plane Poiseuille flow and flat-plate boundary layer flow. This being the case, and also considering the other points discussed above in this paragraph, hence forward we will consider R generically to be a large numerical parameter, and avoid looking at the limit $R \to \infty$. Thus an associated (numerically) small parameter may be defined as ϵ_R, with $\epsilon_R = \frac{1}{R}$.

2.2.2. Quasi-constant or non-constant mean flows

We next look at ‘non-constant’ or ‘quasi-constant’ flows, some associated with the constant flows discussed above and some otherwise. The generic method of stability formulation will be given at the end of this section.

(i) The first example we consider is boundary layer flow over a flat plate. This problem has a unique feature, that, the local Reynolds number R, based on the local boundary layer thickness δ, keeps increasing as the flow moves downstream. Actually $R \sim O(\sqrt{R_x})$ where R_x is the Reynolds number based on x, where x is the distance from the leading edge. In the (elusive) limit $R \to \infty$ the stability problem is given by the

Orr-Sommerfeld equation. However, for engineering problems, since the interest is there for behaviour in the vicinity of the nose of the neutral curve, we will consider that, in the AH methodology, R is a large parameter, but not base everything on $R \to \infty$. Again, as part of the AH methodology, we will at this stage define a small (perturbation) parameter ϵ_N associated with non-constant effects. Basically ϵ_N can be educed from the rate of change of mean-flow with x, and we will say that $\overline{u}_{,x} \sim O(\epsilon_N)$. (Subscript comma (,) followed by a variable (or set of variables) implies derivative, or partial derivative, with respect to that variable (or set of variables)). This implies the following:

$$\frac{\partial}{\partial x} \sim O(\epsilon_N)\,; \qquad \frac{\partial^n}{\partial x^n} \sim O({\epsilon_N}^n)\,. \tag{4}$$

for all slowly varying quantities, like $\alpha_{,x}$ and the amplitude function $\phi(x, y)$, which is defined next. The disturbance streamfunction $\psi(x, y, t)$ may be expressed as follows, for non constant flows:

$$\psi(x, y, t) = \phi(x, y)e^{i[\int \alpha dx - \beta t]}\,, \tag{5}$$

where the wavenumber α is a slowly varying function of x. Also $\phi(x, y)$ is the amplitude function, which again is a a slowly varying function of x. The smallness of $\alpha_{,x}$ and $\phi_{,x}$ are expectedly of the same order as that of $\overline{u}_{,x} \sim O(\epsilon_N)$, so that $\alpha_{,x} \sim O(\epsilon_N)$, and also $\phi_{,x} \sim O(\epsilon_N)$.

Only for the problem under discussion, i.e. for the flat-plate bondary-layer problem ϵ_N can be related to ϵ_R, very formally in the MAE methods, and loosely in the AH methods, as $\epsilon_N \sim O(\epsilon_R)$. So, for $\epsilon_R \to 0$, $\epsilon_N \to 0$. However this kind of relationship is possible to obtain only for rare cases like the problem under discussion. As a generic procedure for the AH methods, (i) one should not look for the limit $R \to \infty$, and (ii) one should not try and relate ϵ_N to ϵ_R; rather keep these in separate compartments. In fact, for an allied problem of flow past an aerofoil, where the length of the plate is finite, one cannot approach the limit $R \to \infty$, although in this problem also, loosely speaking, $\epsilon_N \sim O(\epsilon_R)$. For this problem, as one moves along x, i.e along the length of the plate, R does increase, but the mean flow approaches separation. Hence the limit $R \to \infty$ is not reached. Nevertheless, the stability analysis of this problem is a classical example of success of the PSE method. Perhaps BHS (the proponents of PSE) had it in mind to keep ϵ_R and ϵ_N, in separate compartments; at least they did not try to relate these, nor did they discuss the limiting process of $R \to \infty$.

(ii) The second example is the flat plate boundary layer problem with alternate rigid and porous panels. Initially we have a rigid panel followed by a porous panel where suction equalling $\overline{v} = -\frac{1}{R}$ is applied right away after the junction. Also, R is the local Reynolds number based on the displacement thickness at the end of the rigid plate. This problem is discussed in detail in a companian paper by Paul et al [13](Paper ID 331)

(iii) The third example is that of flow over alternate rigid and porous panels in a parallel channel. This problem is alos discussed in detail in a companian paper by Paul et al [13](Paper ID 331)

2.2.3. The governing equations for non-constant flows

The governing equations are derived next for the non-constant problems introduced above. Before doing that we need to note that that $\alpha_{,x}$ is numerically very small even though $\alpha_{,x} \sim O(\epsilon_N)$. This is true for the flat plate boundary layer case as noted and considered by BHS, and also by us. Also we have noted the same to be true for the two alternate rigid and porous panels problems discussed in (ii) and (iii) above. Keeping this in view we note that the following assumptions may be made; that, $\alpha_{,x}$ is numerically small as compared to $\phi_{,x}$, and that, for higher derivatives in x, $\frac{d^n\alpha}{dx^n}$ are numerically small as compared to $\frac{\partial^n\phi}{\partial x^n}$, at each order of n, for $n \geq 2$. Though not formally necessary to do so, some simplifications may be made in the algebra based on ignoring the derivatives of α with x, on grounds of numerical smallness. Thus in the ensuing derivations eq. (5) is modified to the following:

$$\psi(x, y, t) = \phi(x, y)e^{i[\alpha x - \beta t]}\,. \tag{6}$$

The governing equations, encompassing all the cases (i), (ii), (iii) in subsection 2.2.1, are given as follows.

$$L_{OS}\,\phi + \overline{v}(\phi''' - \alpha^2\phi') + \overline{u}'_{,x}\,\phi = L_2(\phi_{,x})\,, \tag{7a}$$

where the operator L_2 is given as

$$L_2 = -[\overline{u}(D^2 - \alpha^2) - 2\alpha^2(\overline{u} - c) - \overline{u}^{''}] \ . \tag{7b}$$

In the above equation $(7a, b)$, the $\overline{u}, \overline{v}$ fields have to be determined before the stability calculations are performed. For the flat-plate case, corresponding to subsection 2.2.2 para (i), these are calculated from the Blasius solution. For the cases for alternate rigid and porous panels, corresponding to subsection 2.2.2 paras (ii) and (iii), these are calculated as parabolic marching solutions based on standard techniques. Moreover eq. $(7a, b)$ can be given in more compact form as follows:

$$L_T\phi = L_2(\phi_{,x}) \ , \tag{7c}$$

where we may define a non-constant operator L_{NC}, and also the total operator L_T as follows:

$$L_{NC} = [\overline{v}(D^3 - \alpha^2 D) + \overline{u}'_{,x}] \ ; \quad L_T = L_{OS} + L_{NC} \ . \tag{8a, b}$$

The expression $L_T\phi$ has three parameters, β, R, α, of which β is fixed and chosen initially. Here $\overline{v} = -\frac{1}{R}$ for porous channel flow and $\overline{v} = \frac{R_w}{R}$ for boundarylayer flow. Also, at a local station in x, R is known, or R has been chosen as a fixed parameter beforehand. Thus it is α which has to be obtained as part of the solution. To emphasize this fact, sometimes expressions like $L_{OS}\phi$ or $L_T\phi$ will be written as $L_{OS}(\alpha)\phi$ or $L_T(\alpha)\phi$. The right hand side of eq.$(7a, c)$ has been truncated to retain $O(\epsilon_N)$ terms, which is the common practice in PSE, or other non-parallel or non-constant approaches. Should we choose to retain the higher order terms in ϵ_N, we obtain the following equation:

$$L_T(\alpha)\phi = L_2(\phi_{,x}) + L_3(\phi_{,xx}) + L_4(\phi_{,xxx}) \ , \tag{9a}$$

where the operators L_3 and L_4 are given as follows:

$$L_3 = -3i\alpha\overline{u} \ ; \quad L_4 = -\overline{u} \ ; \tag{9b, c}$$

2.3. Rule 1. The rule of additive augmentation

Now, in general an Orr-Sommerfeld like equation can be written as,

$$L(\alpha)\phi = 0 \quad \text{(original)} \quad ; \tag{10}$$

where $L(\alpha)$ is an Orr-Sommerfeld like operator as defined earlier. Let this equation be augmented by another term, so that the augmented equation becomes,

$$L(\alpha)\phi = L_R(\phi) \quad \text{(augmented)} \quad ; \tag{11}$$

where the *largest* order of $L(\alpha)\phi$ is $O(\frac{1}{R})\phi'''' \sim O(R^{\frac{1}{3}})\phi$, using the Tollmien scale, which is adequate for a numerical estimate of orders at a fixed station in x. The size of $L_R(\phi)$ must be less than that of the size of $L_{OS}(\alpha)\phi$. Hence we are all right so far as the size of $L_R(\phi)$ being smaller than $L(\alpha)\phi$ is concerned. (Also subscript R in $L_R(\phi)$ means 'right hand side' term). Suppose that eq. (10) has an eigenvalue $\bar{\alpha}$, and, has a *normalized* eigenfunction, $\overline{\phi}$. The solution for eq. (11) can be obtained as follows:

$$\phi = \bar{\phi} + \phi_f \quad ; \tag{12}$$

where $\overline{\phi}$ is of order $O(1)$, and, $\phi_f \sim o(\phi)$. Also, $\alpha = \bar{\alpha} + \alpha_c$ where α_c is a correction to the eigenvalue $\bar{\alpha}$. Substituting eq. (12) in eq. (11), and remembering that $L(\bar{\alpha})\overline{\phi} = 0$, we have,

$$L(\bar{\alpha})\phi_f = -\alpha_c \frac{\partial}{\partial\alpha}\left[L(\alpha)\overline{\phi}\ \right]_{\alpha=\bar{\alpha}} + L_R\overline{\phi} \quad . \tag{13}$$

From the inviscid part of an Orr-Sommerfeld like equation one obtains,

$$\frac{\partial}{\partial\alpha}\left[L(\alpha)\right]\phi = -iL_2\phi \quad . \tag{14}$$

The solvability condition of eq. (13), after substituting eq. (14) and the definition of $L_2(\phi)$ from eq. (7b) gives,

$$\alpha_c = \frac{\int_0^h \theta\left(L_R\overline{\phi}\right)dy}{\int_0^h \theta\left(iL_2\,\overline{\phi}\right)dy} \quad ; \tag{15}$$

where θ is the adjoint eigenfunction. Also $y = h$ is the outer boundary. Using the Tollmien scale, and remembering the largest size of $L_R(\phi)$ is $\sim \overline{v}\phi'''$, one sees that the largest order of correction that may be encountered in α_c is $O(R^{-\frac{2}{3}})$. Now, we ask, would the same result be obtained if one solves eq. (5) as a homogeneous equation? Namely,

$$[L(\alpha) - L_R]\,\phi = 0 \quad . \tag{16}$$

The new eigenvalue for α, viz. $\bar{\bar{\alpha}}$, for eq. (14), would be,

$$\alpha \rightarrow \bar{\bar{\alpha}} = \bar{\alpha} + \bar{\alpha}_c \quad . \tag{17}$$

Thus, rewriting eq. (16) in terms of the new eigenvalue we obtain,

$$[L(\bar{\bar{\alpha}}) - L_R]\,\bar{\bar{\phi}} = 0 \quad . \tag{18}$$

Also the new eigenfunction is given as, $\phi \rightarrow \bar{\bar{\phi}} = \overline{\phi} + \overline{\phi}_f$. Expansion of eq. (18) gives,

$$L(\bar{\alpha})\bar{\bar{\phi}} + \bar{\alpha}_c\frac{\partial}{\partial\alpha}\left[L(\bar{\alpha})\bar{\bar{\phi}}\right] = L_R\bar{\bar{\phi}} \quad . \tag{19}$$

Again $\overline{\phi}_f$ and $\bar{\alpha}_c$ are small, say of order of some parameter $O(\epsilon) \leq O(R^{-\frac{2}{3}})$ Now simplifying eq. (11) we get:

$$L(\bar{\alpha})\left[\overline{\phi} + \overline{\phi}_f\right] - \alpha_c iL_2\left[\overline{\phi} + \overline{\phi}_f\right] = \epsilon L_R\left[\overline{\phi} + \overline{\phi}_f\right] \quad . \tag{20}$$

Neglecting $O(\epsilon^2)$ terms, we find that eq. (23) becomes,

$$L(\bar{\alpha})\overline{\phi}_f = \bar{\alpha}_c iL_2\overline{\phi} + L_R\overline{\phi} \quad . \tag{21}$$

This eq. (21) is identical to eq. (13), read with eq. (17), and, $\overline{\phi}_f \rightarrow \phi_f$ and $\bar{\alpha}_c \rightarrow \alpha_c$. Hence, when the Orr-Sommerfeld operator is augmented by an operator $L_R\phi$, then there is a correction, α_c, to the eigenvalue which is of order commensurate with the order of $L_R\phi$. This correction is already incorporated in the eigenvalue $\bar{\bar{\alpha}}$, of eq. (21). Equation (18), for the correction to the eigenvalue, consequent to augmentation of the homogeneous equation from eq. (4) to eq. (5), is called "*Rule 1 . The rule of additive augmentation*". It is an important result. The result shows that *terms of different orders can be kept in the same homogeneous equation*, and, the corresponding eigenvalue contains corrections corresponding to the respective different orders of the different terms added. This is one of the main reasons why the AH method works. Therefore, with reference to an earlier discussion above on solution of the OS equation, it is not inconsistent to lump the terms $\frac{1}{R}2\alpha^2\phi''$ and $\frac{1}{R}\alpha^4\phi$, in the same OS equation, and also, it is not inconsistent to keep the terms of $L_{NC}\phi$ (see eqs. $(8a, b)$) in the equation $L_T\phi = 0$.

2.4. Rule 2. The rule of exchange of instability

We now focus attention on eq. $(7c)$. In order to take into account the inhomogeneous $L_2(\phi_{,x})$ term in eq. $(7c)$, we look at a variational form of eq. $(7c)$. Let $\bar{\alpha}$ be the eigenvalue of the homogeneous $L_T(\alpha)\phi = 0$. Further, let the correction in $\bar{\alpha}$, due to the extended eq. $(7c)$, be α_c, so that $\alpha = \bar{\alpha} + \alpha_c$. Thus, eq. $(7c)$ becomes:

$$L_T(\bar{\alpha})\phi + \alpha_c\frac{\partial}{\partial\alpha}\left[L_T(\alpha)\phi\right]_{\alpha=\bar{\alpha}} = L_2(\phi_{,x}) \quad . \tag{22}$$

Also, considering the inviscid part of $L_T(\bar{\alpha})\phi$, we have ,

$$\frac{\partial}{\partial\alpha}\left[L_T(\alpha)\right] = -iL_2 \quad . \tag{23}$$

Equation (25) may now be rewritten as follows:

$$L_T(\bar{\alpha})\phi = i\alpha_c L_2\phi + L_2(\phi_{,x}) \quad . \tag{24a}$$

with the solvability condition given as

$$\int_0^h \theta[i\alpha_c L_2\phi + L_2\phi_{,x}]dy = 0 \quad . \tag{24b}$$

In general there are two kinds of changes in the local eigenfunction ϕ as one moves along x. Between two neighbouring stations in x, one change in ϕ is the change in shape of ϕ. The other change is change in the 'size' of ϕ, or more precisely stated, a change in the norm of ϕ. These two changes can be defined in a precise manner, as follows. The 'size' change part may be defined as an exponential growth (or decay), and the 'shape' change part is what remains after subtracting out the size change part. Hence $L_2(\phi_{,x})$ may be expressed as follows:

$$\phi_{,x} = \sigma\phi + \chi \quad . \tag{25}$$

where, σ is the size change exponent, and, χ, is the shape change part. A proof of the form in eq. (25) is given later. Again the solvability condition of eq. (25), after substituting for $L_2(\phi_{,x})$ from eq. (25), yields the following:

$$\int_0^h \theta[i\alpha_c L_2\phi + \sigma L_2\phi + L_2\chi]dy = 0 \quad . \tag{26}$$

We make the definition of χ precise, by saying that this term is orthogonal to the eigenfunction. Hence we have

$$\int_0^h \theta[L_2\chi]dy = 0 \quad . \tag{27}$$

In view of eqs. (26) and (27), one obtains a very important result, that:

$$\alpha_c = i\sigma \; . \tag{28}$$

The above result is called herein "*Rule 2. The rule of exchange of instability*". If $\sigma > 0$, then ϕ is growing in size, vide eq. (25). However the "correction" $\alpha_c = i\sigma$, which means that *an equal measure* of decay sets in through the term $e^{i\alpha_c x} = e^{i(i\sigma)x} = e^{-\sigma x}$.

The above Rule 2 is only a simple mathematical result, readily understood by a specialist. But it has an important bearing on the versatility in different AH solution procedures, which is an important aspect in AH procedures. Rule 2 also gives rise to the next rule, discussed next.

Here we give a simple proof showing that $\alpha_{,x} \sim O(\overline{u}_{,x}) \sim O(\epsilon_N)$, and $\phi_{,x} \sim O(\overline{u}_{,x}) \sim O(\epsilon_N)$. Proof is also given for the form of $\phi_{,x}$ as given by eq. (28). Differentiating the Orr-Sommerfeld equation ($1a, b$) with respect to x yields the following:

$$L_{OS}(\alpha)\phi_{,x} = -\alpha_{,x} i L_2\phi - i\alpha[\overline{u}_{,x}(D^2 - \alpha^2) - \overline{u}_{,x}'']\phi \; . \tag{29}$$

The solution of the homogeneous equation

$$L_{OS}(\alpha)\phi_{,x} = 0 \; ; \tag{30}$$

gives the 'size' change part of $\phi_{,x}$ (see eq. (28)) as follows:

$$\phi_{,x} = \sigma\phi \; . \tag{31}$$

The solution for the 'shape' change part χ requires the solvability condition

$$\int_0^h \theta\Big\{ - \alpha_{,x} i L_2\phi - i\alpha[\overline{u}_{,x}(D^2 - \alpha^2) - \overline{u}_{,x}'']\phi\Big\}dy = 0 \; ; \tag{32}$$

where from $\alpha_{,x}$ is given as follows:

$$\alpha_{,x} = -\frac{\int_0^h \theta\Big(\alpha[\overline{u}_{,x}(D^2-\alpha^2) - \overline{u}_{,x}{}''] \phi\Big) dy}{\int_0^h \theta\,(L_2\,\phi)\,dy} \quad ; \tag{33}$$

The (numerical) order of ϕ terms in the numerator and denominator of eq. (33) is $\sim O(\phi'')$. Hence it is clear that $\alpha_{,x} \sim O(\overline{u}_{,x})$. Also it is possible to visualise a situation in which the numerical value of $\alpha_{,x}$ is small.

$$\int_0^h \theta\Big(\alpha[\overline{u}_{,x}(D^2-\alpha^2) - \overline{u}_{,x}{}''] \phi\Big) dy \approx 0\,. \tag{34}$$

Further, subject to the solvability condition eq. (33) being satisfied, the right hand side of eq. (29) is orthogonal to the eigensolution (33). Therefore, the inhomogeneous solution of eq. (29), subject to eq. (33) being valid, is given by

$$\phi_{,x} = \chi\,. \tag{35}$$

Further, since the right hand side of eq. (29) is $\sim O(\overline{u}_{,x})$, therefore χ is also $\chi \sim O(\overline{u}_{,x})$. Hence, given eqs. (31) and (35), the form in eq. (25) is obtained.

2.5. Rule 3. The rule of optimal normalization

From the above analysis it is clear that when $\sigma = 0$ then from station to station in x, the ϕ function is *so normalized* that there is no 'size change' in ϕ, and that $L_2(\phi_{,x}) = \chi$ only. Such a normalization, at each station in x, is called the '*optimal normalization*' of ϕ. However, as seen above, when $\sigma \neq 0$, then σ is compensated by α_c. Therefore, as a consequence of optimal normalization of ϕ, the correction $\alpha_c = 0$. Formally therefore the optimal normalization condition is given as

$$\int_0^h \theta[L_2(\phi_{,x})]dy = 0 \quad . \tag{36}$$

In the above equation $L_2(\phi_{,x})$ may be obtained numerically as a backward difference with respect to the current station, and the normalization of ϕ could be set at the current station by use of eq. (29) above.

2.6. Summary of AH methodlogy

We need to solve,

$$L_T(\alpha)\phi = L_2(\phi_{,x})\,, \tag{37}$$

Method 1:

We solve the homogeneous part to obtain α, and α_i gives the best measure of th espatial growth rate as this is not associated with nay monitored property of the eigensolution, e.g. the inner maximum of ϕ. We use the method of optimal normalisation to acccount for the right hand side of the above equation (31).

There are two other broad approaches for solution based on AH methods. These are the following:

Method 2:

Solve homogeneous part of $L_T(\alpha)\phi = 0$. Then choose any suitable normalisation for ϕ, say, keeping the inner maximum of the disturbance root mean squared velocity, u_{rms}, constant at each station in x. Thereafter, at each station in x determine the correction in the eigenvalue $\bar{\alpha}$, i.e α_c, by using eq. (27). Gaster[1] uses this approach.

Method 3:

Consider the equation below:

$$L_T(\alpha)\phi = L_2(\phi_{,x})\,, \tag{7c \& 37}$$

Solve the inhomogeneous equation directly, at each station in x, using any suitable normalisation for ϕ, say, keeping the inner maximum of the disturbance root mean squared velocity, u_{rms}, constant at each station in x. This inhomogeneous solution gives the corrected value of α, i.e. $\alpha = \bar{\alpha} + \alpha_c$, at each station in x. BHS and GN use this method. Further, those AH methods that have subjective definitions of the norm of ϕ without fitting it into one or other of the three procedures discussed above, do not handle the $\phi_{,x}$ term appropriately. Therefore, these methods get incorrect numerical results that do not match with DNS results. These methods are wrong.

3. Results and Discussions

First of all we re-solved the BHS problem and the GN problem for the flat plate boundary layer problem using the method of optimal normalisation, because this latter method has not been defined and used in earlier works. This is in the (F, R)-plane where F is the frequency parameter, $F = \left(\frac{\beta}{R}\right) \times 10^6$, and R is the Reynolds number based on the momentum thickness. The results are compared with those of Gaster [1], Govindarajan and Narasimha [5], and Fasel and Konzelmann [6]. The discussion is mainly to highlight that the eigenvalue corresponding to the homogeneous equation ($12a$) does not give a measure of overall growth rate in any sense. This is because the form of $L_T(\alpha)\phi$ depends, amongst other things, on the coordinate scaling used.

4. Conclusions

The Gaster and GN methods are restricted to the flat plate boundary layer problem. But, the PSE method of BHS is very general and has wide applicability so long as ϵ_R and ϵ_N are small, and ϵ_R and ϵ_N are notionally kept in separate compartments. The PSE method has been extended by BHS to non-linear problems as well, after introducing a (numerically) small, though finite amplitude, parameter ϵ_A, for the amplitude of ϕ. BHS calculate the different harmonic level contributions, due to non-linearity, and these correspond to different powers of ϵ_A. Thereafter, all the terms, in different orders of ${\epsilon_A}^n$, corresponding to distortion of the fundamental, are lumped together and added to the right hand side of eq. ($7c$ & 37). Justification and reprieves? Rule 1 read with Rules 2 and 3. The results of non-linear analysis by BHS, using their extended version of PSE, compare well with the results of Sen and Vashist [14] who use the Stuart -Watson amplitude expansion methodology for the same problem, using the Shanks [15] method for accelerated convergence of the Stuart-Landau series.

At the end of this paper we note that the flexibility and beauty of the PSE method is amazing, and that, there are no grounds to fault the mathematical rigour in this method, given Rules 1, 2 and 3.

The Gaster and GN methods also ‘work’ for the flat plate boundary layer problem. These two approaches have some pretensions of being asymptotic theories for $R \to \infty$. On that count these are unsuccessful. The only proper asymptotic theories for the flat plate boundary layer problem are those based on MAE, Smith[16] and Smith and Bodonyi[17]. However, the Gaster and GN methods ‘work’, not because their asymptotics is proper, but because of Rules 1, 2 and 3.

This paper is dedicated to Professor Michael Gaster F.R.S., who started it all, and got the first correct answers for the flat plate boundary layer problem.

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